Inflationary spectra, decoherence, and two-mode coherent states

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Abstract

We re-examine the question of the entropy stored in the distribution of primordial density fluctuations. To this end we make use of two-mode coherent states. These states incorporate the isotropy of the distribution as well as the temporal coherence and the semi-classical character of highly amplified modes. They also provide a lower bound for the entropy if, as one expects, decoherence processes erase the squeezing which originally characterized the distribution in inflationary models. This lower bound is one half the maximal (thermal) value. By considering backreaction effects, we also provide an upper bound for this entropy at the onset of the adiabatic era.

I. INTRODUCTION

Inflation tells us that the primordial density fluctuations arise from the amplification of vacuum fluctuations [1, 2]. As a result of this amplification, the initial vacuum state becomes a product of highly squeezed two-mode states [3]. In spite of the complexity of this state, when computing expectation values, i.e. the two-point function, the modes exhibit a temporal coherence upon horizon re-entry. When considering the physics which took place near the recombination, it is therefore convenient and sufficient to enforce the temporal coherence by putting to zero the decaying mode. Then the residual random properties consist in treating the amplitude of the growing mode as a stochastic variable, thereby ignoring the quantum properties of the original Gaussian distribution.

However this simplified description has several drawbacks. In particular the settings are too restrictive to describe the distribution which would result from some decoherence process which would have taken place during the early universe. More generally, the simplified settings are unable to parameterize deviations from the standard results which preserve the isotropy and the Gaussianity of the distribution.

In this paper, we show that the appropriate basis to investigate these questions is provided by two-mode coherent states. The reasons are the following. First, a two-mode

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coherent state provides the quantum counterpart of a particular classical realization of the ensemble of metric fluctuations. This correspondence is well defined for the highly excited modes we are dealing with. (Remember that the observed temperature anisotropies of relative amplitude 10^{-5} fix the occupation number n to be of the order of 10^{100}.) Second, ensembles of two-mode coherent states can characterize *any* Gaussian isotropic distribution, thereby allowing to describe arbitrary levels of coherence. This follows from the requirement of isotropy and homogeneity which restricts the non-vanishing matrix elements of the distribution: only two-mode elements with opposite wave vectors could be different from zero. Third, they allow to make contact with the general remark of [4, 5] according to which squeezed states rapidly decay into a statistical mixture of coherent states when small non-linearities are no longer neglected[6]. In fact when applying this general theorem to highly squeezed two-mode states, the resulting distribution is precisely a diagonal density matrix of two-mode coherent states.

This distribution defines the *minimal* entropy stored in the primordial spectrum (or in other words, the minimal coarse graining) when no squeezing remains, that is, when there is no direction in phase space in which the spread of the non-diagonal elements of the density matrix is smaller than that of the vacuum. For each mode the entropy bound is $1/2$ the maximal (thermal) value, given the occupation number. We shall see that this entropy coincides with that associated with the simplified prescription which consists in neglecting the decaying mode. It is to be stressed that this identification can be reached because in the large occupation number limit, the entropy is extremely sensitive to the level of decoherence whereas the power spectrum is instead extremely robust. Typically the relative modifications of the latter are $1/n$ whereas the changes of the former are in $\ln n$.

The open question concerns the efficiency of decoherence processes in nature: are they powerful enough to suppress the squeezing that the initial density matrix possessed? This question is currently under investigation. Preliminary results indicate that the squeezing is indeed erased, thereby implying that the resulting entropy is larger than (or equal to) the above mentionned bound.

Finally, we shall also provide an upper bound for this entropy by considering backreaction effects at the end of the inflationary period. This upper bound is given by $3/4$ of the maximal entropy. The evaluation of these entropies is exactly performed by exploiting the fact that any Gaussian isotropic distribution can be expressed in terms of thermal distributions, see Appendix C.

Related questions have been already analyzed in several papers, see [7–13]. What we add in the present paper is a further clarification of the matters, the usefulness and the relevance of two-mode coherent states, the lower and upper bounds on the entropy, and relationships between various elements which have been some how separately discussed. Notice that we shall not discuss the physical relevance of this entropy for structure formation. For this interesting question we refer to [7].

II. A REVIEW OF THE STANDARD DERIVATION OF PRIMORDIAL SPECTRA

A. Quantum distribution of two-mode states

In this subsection we recall how the amplification of modes of a massless field propagating in a FRW universe translates in quantum settings in the fact that the initial ground state evolves into a product of highly squeezed two-mode states. Before proceeding, we remind the reader that it has been shown that the evolution of linearized cosmological perturbations (metric and density perturbations) reduces to the propagation of massless scalar fields in a FRW spacetime [14]. In this article, we shall only consider the massless scalar test field since the transposition of the results to physical fields represents no difficulty. Indeed, when preserving the linearity of the evolution, the only modification concerns the late time dependence of the modes.

Let us work in a flat FRW universe. The line element is:

$$ds^2 \;=\; a(\eta)^2 \left[-d\eta^2 + \delta_{ij} dx^i dx^j\right] \,. \tag{1}$$

For definiteness and simplicity, we consider a cosmological evolution which starts with an inflationary de Sitter phase and ends with a radiation dominated period. When using the conformal time η to parametrize the evolution, the scale factor is respectively given by

$$a(\eta) = -\frac{1}{H(\eta - 2\eta_r)}\,, \qquad \text{for} \quad -\infty < \eta < \eta_r\,, \tag{2a}$$

$$a(\eta) = \frac{1}{H\eta_r^2}\,\eta\,, \qquad \text{for} \quad \eta > \eta_r\,, \tag{2b}$$

where $\eta_r > 0$ designates the end of inflation. The transition is such that the scale factor and the Hubble parameter are continuous functions. This approximation based on an instantaneous transition is perfectly justified for modes relevant for CMB physics. Indeed, their wave vector k obeys $k\eta_r \sim 10^{-25} \sim e^{-60}$ when inflation lasts for 60 e-folds. Hence, the phase shift they could accumulate in a more realistic smoothed out transition is completely negligible.

Let $\xi(\eta, \mathbf{x})$ be a massless test scalar field propagating in this background metric. It is convenient to work with the rescaled field $\phi = a\xi$ and to decompose it into Fourier modes

$$\phi(\eta, \mathbf{x}) \;=\; \int d^3k\, \frac{e^{i\mathbf{k}\mathbf{x}}}{(2\pi)^{3/2}} \phi_{\mathbf{k}}(\eta)\,. \tag{3}$$

The time dependent mode $\phi_{\mathbf{k}}$ obeys

$$\partial_\eta^2 \phi_{\mathbf{k}} + \left(k^2 - \frac{\partial_\eta^2 a}{a}\right) \phi_{\mathbf{k}} = 0\,, \tag{4}$$

where $k = |\mathbf{k}|$ is the norm of the conformal wave vector.

In our background solution, $k^2 - \partial_\eta^2 a/a$ is negative during the de Sitter period when the wavelength is larger that the Hubble radius. This leads to a large amplification of ϕ_k. In quantum terms this mode amplification translates into spontaneous pair production characterized by correspondingly large occupation numbers.

To obtain the final distribution of particles, one should introduce two sets of positive frequency solutions of Eq. (4). The in modes are defined at asymptotic early time, and the out ones at late time. Both have unit positive Wronskian in conformity with the usual particle interpretation [15]. One gets

$$\phi_k^{in}(\eta) = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k(\eta - 2\eta_r)}\right) e^{-ik(\eta-2\eta_r)}\,, \quad \text{for} \quad \eta < \eta_r\,, \tag{5a}$$

$$\phi_k^{out}(\eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta}\,, \quad \text{for} \quad \eta > \eta_r\,. \tag{5b}$$

In spite of the time dependence of the background, these two positive frequency modes are unambiguously defined (up to an arbitrary constant phase which drops in all expectation values and which has here been chosen so as to simplify the forthcoming expressions). In the radiation dominated era there is no ambiguity since the conformal frequency is constant because $\partial_\eta^2 a = 0$. In the de Sitter epoch, there is no ambiguity either for relevant modes if inflation lasts more than 70 e-folds, see [16] for the evaluation of the small corrections one obtains when imposing vacuum at some finite early time. Similarly, in quantum settings, there is no ambiguity for the initial state of relevant modes: at the onset of inflation they must be in their ground state[14, 17].

The in and out modes are related by a Bogoliubov transformation

$$\phi_k^{in}(\eta) = \alpha_k \phi_k^{out}(\eta) + \beta_k^* \phi_k^{out\,*}(\eta)\,, \tag{6}$$

where the Bogoliubov coefficients are given by the Wronskians

$$\alpha_k = \left(\phi_k^{out}, \phi_k^{in}\right)\,, \quad \beta_k^* = -\left(\phi_k^{out\,*}, \phi_k^{in}\right)\,. \tag{7}$$

These overlaps should be evaluated at transition time η_r since modes satisfy different equations in each era. One gets

$$\alpha_k = -\frac{e^{2ik\eta_r}}{2k^2\eta_r^2}\left(1 - 2ik\eta_r - 2k^2\eta_r^2\right) = \frac{-1}{2k^2\eta_r^2}\left(1 + O(k\eta_r)^3\right)\,, \tag{8a}$$

$$\beta_k^* = \frac{1}{2k^2\eta_r^2}\,. \tag{8b}$$

Thus, for relevant modes, $k\eta_r \sim 10^{-25}$, the in modes are enormously amplified. Concomitantly, they are dominated by the sine during the radiation dominated era:

$$\phi_k^{in}(\eta) \;=\; \frac{i}{k^2\eta_r^2}\left[\frac{\sin k\eta}{\sqrt{2k}} + O(k\eta_r)^3 \cos k\eta\right]\,, \qquad \eta \geq \eta_r\,. \tag{9}$$

Once the cosine in Eq. (9) is neglected, the physical modes ϕ_k^{in}/a show the same temporal behaviour, e.g. they are constant until they start oscillating as they re-enter the Hubble radius when $k\eta \simeq 1$.

Lets now see how these considerations translate in second quantized settings. Each mode operator is decomposed twice

$$\hat\phi_{\mathbf{k}}(\eta) = \hat a_{\mathbf{k}}^j \phi_k^j(\eta) + \hat a_{-\mathbf{k}}^{j\,\dagger} \phi_k^{j\,*}(\eta)\,, \tag{10}$$

where j stands for both the in and out basis. The operators so defined are related by the transformation

$$\hat{a}_{\mathbf{k}}^{in} = \alpha_k^* \, \hat{a}_{\mathbf{k}}^{out} - \beta_k \, \hat{a}_{-\mathbf{k}}^{out\,\dagger} \,. \tag{11}$$

This transformation couples $\mathbf{k}$ to $-\mathbf{k}$ only. Hence, when starting from the in vacuum (the state annihilated by the $\hat{a}_{\mathbf{k}}^{in}$ operators), every out particle of momentum $\mathbf{k}$ will be accompanied by a partner of momentum $-\mathbf{k}$. Moreover, pairs characterized by different momenta are incoherent (in the sense that in expectation values any product of annihilation and creation operators of different momenta will factorize).

These two properties are explicit when writing the in vacuum in terms of out states (i.e. states with a definite out particle content). From Eq. (11), one gets (see [7, 19])

$$\begin{aligned} |0,\, in\rangle &= \widetilde{\prod_{\mathbf{k}}} \otimes |0,\, \mathbf{k}, in\rangle_2 \\ &= \widetilde{\prod_{\mathbf{k}}} \otimes \left(\frac{1}{|\alpha_k|} \exp\left(z_k \, \hat{a}_{\mathbf{k}}^{out\,\dagger} \, \hat{a}_{-\mathbf{k}}^{out\,\dagger} \right) |0,\, \mathbf{k}, out\rangle \otimes |0,\, -\mathbf{k}, out\rangle \right) . \end{aligned} \tag{12}$$

The tilde tensorial product takes into account only half the modes. It must be introduced because the squeezing operator acts both on the $\mathbf{k}$ and the $-\mathbf{k}$ sectors. The definition of this product requires the introduction of an arbitrary wave vector to divide the modes into two sets. The sign of k_z can be used. Notice that a rigorous definition of $\widetilde{\prod_{\mathbf{k}}}$ requires to consider a discrete set of modes normalized with Kronekers (that is, to normalize the modes in a finite conformal 3-volume). To be explicit, the two-mode state $|0,\, \mathbf{k}, in\rangle_2$ is given by

$$|0,\, \mathbf{k}, in\rangle_2 = |0,\, \mathbf{k}, in\rangle \otimes |0,\, -\mathbf{k}, in\rangle \,, \tag{13}$$

where $|0,\, \mathbf{k}, in\rangle$ is the ground state of the $\mathbf{k}$-th mode at the onset of inflation. The complex parameter z_k appearing in the squeezing operator in Eq. (12) is given by the ratio of the Bogoliubov coefficients

$$\begin{aligned} z_k &= \frac{\beta_k}{\alpha_k^*} = -e^{-2ik\eta_r} \frac{1}{(1 + i2k\eta_r - 2k^2\eta_r^2)} \\ &= -1 + O(k\eta_r)^3 \,. \end{aligned} \tag{14}$$

The high occupation number limit corresponds to $|z_k| \to 1^-$.

It has to be emphasized that none of the out states in Eq. (12) carry 3-momentum. Hence, the distribution is homogeneous in a strong sense: at late time the 3-momentum operator is still annihilated by the state of Eq. (12). (This property is not satisfied by incoherent distributions such as thermal baths. In those cases, the 3-momentum fluctuates and vanishes only in the mean.) The present distribution is also isotropic since the Bogoliubov coefficients are functions of the norm k only. Finally, it is a Gaussian distribution, as can be seen from Eq. (12).

To appreciate the peculiar properties of the distribution of Eq. (12) it is interesting to consider the most general homogeneous, isotropic, and Gaussian distribution of out

quanta. Its properties are completely specified by three real functions of the norm k (one real and one complex) through the following expectation values

$$\langle \hat{a}_{\mathbf{k}}^{out} \rangle = 0\,, \tag{15a}$$

$$\langle \hat{a}_{\mathbf{k}}^{out\,\dagger}\, \hat{a}_{\mathbf{k}'}^{out} \rangle = n_k\, \delta^3(\mathbf{k} - \mathbf{k}')\,, \tag{15b}$$

$$\langle \hat{a}_{\mathbf{k}}^{out}\, \hat{a}_{\mathbf{k}'}^{out} \rangle = c_k\, \delta^3(\mathbf{k} + \mathbf{k}')\,. \tag{15c}$$

In the second line, n_k is the mean occupation number. In the third one, the complex number c_k characterizes the quantum coherence of the distribution. The degree of two-mode coherence is given by $|c_k|/(n_k + 1/2)$, see Appendix C. It is bounded by 1. For a thermal (incoherent) distribution, one has $c_k \equiv 0$: no coherence.

In the case of pair production from vacuum, one has

$$\begin{aligned} n_k &= |\beta_k|^2 = \frac{|z_k|^2}{1 - |z_k|^2}\,, \\ c_k &= \alpha_k \beta_k = \frac{z_k}{1 - |z_k|^2}\,. \end{aligned} \tag{16}$$

Therefore, when considering relevant modes in our inflationary model Eq. (8), one has

$$\begin{aligned} n_k &= \frac{1}{4(k\eta_r)^4} \simeq 10^{100}\,, \\ \frac{|c_k|}{n_k + 1/2} &= 1 + O(k\eta_r)^3\,. \end{aligned} \tag{17}$$

That is, the distribution which results from inflation is highly populated and, more importantly, maximally coherent. When computing the Green function, the two-mode coherence of the distribution will manifest itself in the *temporal coherence* of the modes.

B. Two-point function and the neglect of the decaying mode

When expressed in terms of out modes, the two-point function associated with the general distribution specified by Eqs. (15) is

$$\begin{aligned} G(\eta, \mathbf{x}, \eta', \mathbf{x}') &= G_{out}(\eta, \mathbf{x}, \eta', \mathbf{x}') \\ &+ \int_0^\infty \frac{dk k^2}{\pi^2} \frac{\sin(k|\mathbf{x} - \mathbf{x}'|)}{k|\mathbf{x} - \mathbf{x}'|} 2Re\left[n_k\, \phi_k(\eta)\phi_k^*(\eta') + c_k\, \phi_k(\eta)\phi_k(\eta') \right]. \end{aligned} \tag{18}$$

In the first line, we have isolated G_{out}, the contribution of the out vacuum. In the high occupation number limit, this contribution is negligible.

It is important to notice that, for a general distribution, the sum in bracket in the above integrand cannot be factorized. However it does factorize in two cases which are relevant for us: first, for coherent states, see Appendix A, and second for distributions resulting from pair production from vacuum. Indeed, when considering our cosmological model, taking into account the minus sign of Eq. (14), and neglecting correction terms in $O(k\eta_r)^3$ (which amounts to neglect the decaying mode, see Eq. (9)), one obtains

$$G_{in}(\eta, \mathbf{x}, \eta', \mathbf{x}') = \int_0^\infty \frac{dk k^2}{\pi^2} \frac{\sin(k|\mathbf{x} - \mathbf{x}'|)}{k|\mathbf{x} - \mathbf{x}'|}\, n_k\, \frac{\sin k\eta}{\sqrt{k}} \frac{\sin k\eta'}{\sqrt{k}}\,. \tag{19}$$

As announced, the integrand in Eq. (18) factorizes. In inflation, the function which appears is $\sin k\eta$ where η is related to the scale factor by Eq. (2). This is how the temporal coherence of modes obtains from the two-mode coherence of the distribution.

Once the cosine is neglected, the quantum distribution is effectively replaced by a stochastic Gaussian distribution of classical fluctuations

$$\phi_{\mathbf{k}}(\eta) = S_{\mathbf{k}} \frac{\sin k\eta}{\sqrt{k}} \,, \tag{20}$$

with locked temporal argument, and random amplitudes with variances given by

$$\langle\!\langle\, S_{\mathbf{k}} S^*_{\mathbf{k}'} \,\rangle\!\rangle_{eff} = \langle\!\langle\, S_{\mathbf{k}} S_{-\mathbf{k}'} \,\rangle\!\rangle_{eff} = 2n_k \, \delta^{(3)}(\mathbf{k} - \mathbf{k}') \,. \tag{21}$$

Being Gaussian, the effective probability distribution is simply

$$\mathcal{P}_{eff} = \widetilde{\prod_{\mathbf{k}}} \frac{1}{2n_k} \exp\left(- \frac{|S_{\mathbf{k}}|^2}{2n_k} \right) \,. \tag{22}$$

To avoid double counting, one must again use the tilde product which takes into account half the modes only, as was done in the quantum distribution of Eq. (12). This counting becomes crucial when computing the entropy, see Section III.C.

Notice finally that the first equality in Eq. (21) is simply the expression of the reality of the field $\phi(\eta, \mathbf{x})$. This was not the case when noticing that inflation gives $\langle \hat{a}^{out\,\dagger}_{\mathbf{k}} \, \hat{a}^{out}_{\mathbf{k}'} \rangle = -\langle \hat{a}^{out}_{\mathbf{k}} \, \hat{a}^{out}_{-\mathbf{k}'} \rangle (1 + O(k\eta_r)^3)$ for the quantum field operators. In that case the equality is the expression of the coherence of the in vacuum, i.e. the absence of 3-momentum fluctuations and the possibility of factorizing the 2-point function by discarding the cosines.

C. Additional remarks

First we remind the reader why a random distribution of both the sine and the cosine does not give rise to any temporal coherence [3]. In fact such a distribution corresponds to an incoherent (thermal) distribution.

Writing the field modes as the sum of a sine and a cosine:

$$\begin{aligned} \hat{\phi}_{\mathbf{k}} &= \hat{a}^{out}_{\mathbf{k}} \phi^{out}_k + \hat{a}^{\dagger\, out}_{-\mathbf{k}} \phi^{out\,*}_k \\ &= \hat{C}_{\mathbf{k}} \frac{\cos(k\eta)}{\sqrt{k}} + \hat{S}_{\mathbf{k}} \frac{\sin(k\eta)}{\sqrt{k}} \,. \end{aligned} \tag{23}$$

It is to be noticed that the operators $\hat{C}_{\mathbf{k}}$ and $\hat{S}_{\mathbf{k}}$ are proportional to the field mode operator and its conjugate momentum evaluated at $\eta = 0$, as if there were no inflation:

$$\begin{aligned} \hat{C}_{\mathbf{k}} &= \frac{1}{\sqrt{2}} \left(\hat{a}^{out}_{\mathbf{k}} + \hat{a}^{\dagger\, out}_{-\mathbf{k}} \right) = \sqrt{k} \, \hat{\phi}_{\mathbf{k}}(0) \,, \\ \hat{S}_{\mathbf{k}} &= \frac{-i}{\sqrt{2}} \left(\hat{a}^{out}_{\mathbf{k}} - \hat{a}^{\dagger\, out}_{-\mathbf{k}} \right) = \frac{1}{\sqrt{k}} \, \partial_\eta \hat{\phi}_{\mathbf{k}}(0) \,. \end{aligned} \tag{24}$$

They satisfy the canonical equal time commutation relations.

Consider an incoherent ($c_k = 0$) distribution:

$$\langle \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}'} \rangle_{inc} = n_k \, \delta^3(\mathbf{k} - \mathbf{k}') \,, \qquad \langle \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}'} \rangle_{inc} = 0 \,. \tag{25}$$

Then $\hat{C}_{\mathbf{k}}$ and $\hat{S}_{\mathbf{k}}$ are uncorrelated Gaussian operators with equal variance:

$$\begin{aligned} \langle \hat{C}_{\mathbf{k}} \hat{C}^\dagger_{\mathbf{k}'} \rangle_{inc} &= \langle \hat{S}_{\mathbf{k}} \hat{S}^\dagger_{\mathbf{k}'} \rangle_{inc} = \left(n_k + \frac{1}{2} \right) \delta^3(\mathbf{k} - \mathbf{k}') \,, \\ \langle \hat{C}_{\mathbf{k}} \hat{S}_{\mathbf{k}'} \rangle_{inc} &= 0 \,. \end{aligned} \tag{26}$$

This implies the absence of temporal coherence of the modes, as can be seen from the temporal behaviour of the bracket in the integrand of Eq. (18): when $\eta = \eta'$ the bracket does not exhibit any oscillation in k as it did for the inflationary distribution. (This absence can also be understood by considering the $S_{\mathbf{k}}$ and $C_{\mathbf{k}}$ as stochastic variables rather than quantum ones. Writing the mode in terms of its norm and its phase [3, 20]

$$\phi_{\mathbf{k}} = \Phi_{\mathbf{k}} \sin(k\eta + \theta_{\mathbf{k}}) \,. \tag{27}$$

one can treat $\Phi_{\mathbf{k}}$ and $\theta_{\mathbf{k}}$ as stochastic variables. One then verifies that the distribution for the phase $\theta_{\mathbf{k}}$ is uniform over the interval $[0, 2\pi]$. Hence no temporal coherence could obtain.)

As a last remark, to estimate what has been neglected when discarding the cosine in Eq. (20), we compute the residual fluctuations of $\hat{C}_{\mathbf{k}}$ and the cross correlation $\hat{C}_{\mathbf{k}} \hat{S}_{\mathbf{k}}$. To leading order in n_k, one has

$$\begin{aligned} \langle \hat{S}_{\mathbf{k}} \hat{S}^\dagger_{\mathbf{k}'} \rangle_{in} &= \left(n_k + \frac{1}{2} - Re(c_k) \right) \delta^3(\mathbf{k} - \mathbf{k}') = 2 n_k \, \delta^3(\mathbf{k} - \mathbf{k}') \,, \\ \langle \hat{C}_{\mathbf{k}} \hat{C}^\dagger_{\mathbf{k}'} \rangle_{in} &= \left(n_k + \frac{1}{2} + Re(c_k) \right) \delta^3(\mathbf{k} - \mathbf{k}') = O(n_k^{1/4}) \, \delta^3(\mathbf{k} - \mathbf{k}') \,, \\ \langle \hat{C}_{\mathbf{k}} \hat{S}^\dagger_{\mathbf{k}'} \rangle_{in} &= - \left(Im(c_k) + \frac{i}{2} \right) \delta^3(\mathbf{k} - \mathbf{k}') = O(n_k^{1/4}) \, \delta^3(\mathbf{k} - \mathbf{k}') \,. \end{aligned} \tag{28}$$

When divided by the variance of $\hat{S}_{\mathbf{k}}$, the variance of $\hat{C}_{\mathbf{k}}$ and the cross-correlation are of order $n_k^{-3/4} \sim (k\eta_r)^3 \sim 10^{-75}$. One can therefore safely use the distribution Eq. (22) in replacement of the quantum distribution Eqs. (12) or (28) when calculating the power spectrum. However, in general, this is *not* the case for the entropy.

D. Drawbacks of the simplified description

The simplified description in terms of a statistical ensemble of sine standing waves has indeed several shortcomings. It is of value to describe them with some attention.

First, it should be pointed out that in the early universe, different physical processes could give rise to different final density matrices, and hence, to different entropies. As we shall see bellow, only a narrow set of these quantum distributions can be put in correspondence with that of Eq. (22). To describe the most general isotropic and Gaussian distribution, it is necessary to return to two-mode distributions characterized by n_k and c_k

of Eq. (15). Second, the classical ensemble of sine waves should be considered only as an effective description of the density matrix. The appropriate basis to describe this matrix is provided by coherent states. The reasons are the following. On the one hand, they constitute the quantum counterparts of classical configurations in phase space. Therefore they provide an adequate basis for studying the semi-classical limit. In particular, since the spread of coherent states have no preferred direction in phase space, it will be easy to see whether or not a distribution has kept some squeezing. On the other hand, they are the preferred basis in which squeezed states decohere when weak interactions are taken into account[5].

III. TWO-MODE COHERENT STATES

When using coherent states in cosmology, one must pay attention to the entanglement between $\mathbf{k}$ and $-\mathbf{k}$ modes. A naive use of coherent states which would assign amplitudes to each mode separately could erase these correlations and therefore suppress the information about the temporal phase of the modes. Taking into account the entanglement leads to the notion of 'two-mode coherent states'.

A. Representation of the in vacuum with two-mode coherent states

To understand the usefulness of two-mode coherent states it is appropriate to first mention the following properties [21]. Consider a mode $\mathbf{k}$ in a coherent state $|v,\, \mathbf{k}\rangle$ constructed with out operators, see Appendix A for its definition. Then compute the one-mode reduced state obtained by projecting it on the two-mode initial vacuum of Eq. (13)

$$\langle v,\, \mathbf{k}|0in,\, \mathbf{k}\rangle_2 = \mathcal{A}_k(v)\, |z_k v,\, -\mathbf{k}\rangle\,. \tag{29}$$

It is remarkable that the state of the $-\mathbf{k}$ mode which is entangled to $|v,\, \mathbf{k}\rangle$ is also a coherent state. Its amplitude is given by the complex conjugate of v times z_k characterizing the pair creation process. These facts follow from the EPR correlations in the initial vacuum displayed in Eq. (12). The prefactor $\mathcal{A}_k(v)$ is

$$\mathcal{A}_k(v) \;=\; \frac{1}{|\alpha_k|} \exp\left(-\frac{|v|^2}{2|\alpha_k|^2} \right). \tag{30}$$

It is the probability amplitude to find the mode $\mathbf{k}$ in the coherent state $|v,\, \mathbf{k}\rangle$ given that we start with vacuum at the onset of inflation.

Using the representation of the identity with coherent states, Eq. (A13), the two-mode in vacuum can be thus decomposed as a *single* sum of *two-mode coherent states* rather than two independent integrations over one-mode coherent states. In fact we have

$$|0in,\, \mathbf{k}\rangle_2 = \int \frac{d^2v}{\pi}\, \mathcal{A}_k(v)\; |v,\, \mathbf{k}\rangle \otimes |z_k v^*,\, -\mathbf{k}\rangle\,. \tag{31}$$

This result is exact[1] and applies even for low occupation numbers. It is the consequence of the coherence of the in vacuum and holds for every homogeneous pair creation process.

Another important consequence of Eq. (29) is that the probability to find *simultaneously* the **k**-mode with coherent amplitude v and its partner with amplitude w is

$$\mathcal{P}_{2,\,k}(v,w) = |\langle v,\,\mathbf{k}|\langle w,\,-\mathbf{k}|0in,\,\mathbf{k}\rangle_2|^2 = |\mathcal{A}_k(v)|^2 \times e^{-|w\,-\,z_k v^*|^2}\,. \tag{32}$$

The second factor arises follows from the overlap between two different coherent states: $|\langle u|v\rangle|^2 = \exp(-|u-v|^2)$. Equation (32) implies that once the amplitude of the **k**-mode has been measured, the conditional probability to find its partner in a coherent state $|w\rangle$ is centered around $w = z_k v^*$. In the high occupation number limit we are dealing with, the spread $(= 1)$ around this mean value is negligible when compared to the spread in v which is given by $|\alpha_k|^2 = n_k + 1$. Therefore, when computing expectation values in leading order in n_k, the conditional probability acts as a delta function on both the real and the imaginary part of w. This is how the EPR correlations in the in-vacuum translate in the coherent states basis. This result determines the properties of the local correlations in the primordial spectra[21].

To complete this analysis, and in preparation for studying decoherence, it is also interesting to explicitly write the non-diagonal matrix elements of the in vacuum density matrix. One has

$$\langle v|\langle w|\hat{\rho}_{in}|v'\rangle|w'\rangle \;=\; \mathcal{A}_{2,\,k}(v,w)\,\mathcal{A}_{2,\,k}(v',w')^*\,, \tag{33}$$

where the two-mode amplitude is

$$\mathcal{A}_{2,\,k}(v,w) \;=\; \mathcal{A}_k(v)\,e^{-\frac{1}{2}|w\,-\,z_k v^*|^2}\,e^{i\,\mathrm{Im}(w^* z_k v^*)}\,. \tag{34}$$

Since the initial vacuum is a pure state and since $|\alpha_k|^2 \gg 1$, the above matrix elements do not vanish, *even for macroscopically different coherent states.* Therefore this distribution does not describe a classical ensemble of these quasi-classical states. Fortunately, such quantum distributions are unstable to any weak perturbation in that they rapidly evolve into statistical mixtures. Let us now describe this decoherence process.

B. Zurek *et al.* analysis and minimal decoherence scheme

In general, it is a difficult question to determine into what mixture an initial density matrix will evolve when taking into account some interactions amongst modes or with other modes. There exist however several cases where clear conclusions can be drawn. First, when one can neglect the free Hamiltonian, the preferred states (that is the basis into which the reduced density matrix will become diagonal) are the eigenstates of the interaction Hamiltonian [22–24]. This approach has been applied in [12], to primordial density fluctuations when the (physical) modes are almost constant because their wave

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[1] Notice however that the above decomposition is not unique since the coherent states are not orthogonal, compare with [27]. Eq. (31) has the advantage to be directly related to the detection of a quasi-classical configuration in the **k** sector.

length is much larger than the Hubble radius. The conclusion is that the preferred basis is provided by amplitude (position) eigenstates. However this conclusion leaves some ambiguity and might lead to some difficulties. First, position eigenstates are not normalized. Second, and more importantly, the spread in momentum is infinite for these states. Therefore, the velocity field would not be well defined when the modes re-enter the horizon. Moreover, as pointed out in [13], some additional decoherence could be obtained as they re-enter the horizon. In this case, the momentum should be treated in the same footing as the position. To cure these problems, some finite spread in position should be introduced. One then needs a physical criterion to choose this spread.

To remove this ambiguity it is appropriate to appeal to coherent states both for mathematical and physical reasons. In this article, we shall only present the basic mathematical results. We reserve for a forthcoming publication the justification of the physical relevance of these states in a cosmological context. Let us simply notice the following points. In inflationary cosmology, modes are weakly interacting harmonic oscillators[6]. Indeed, given that the relative density fluctuations have small amplitude ($\sim 10^{-5}$), the hypothesis of weak interactions is perfectly legitimate. Second, coherent states provide the basis in which the density matrix decoheres when considering weakly interacting harmonic oscillators. This has been shown by Zurek & *al.* [5]. The criterion they used to reach this conclusion is the minimization of the growth of entropy in the course of the evolution. With this criterion, coherent states are more stable than squeezed states in that the growth of entropy one obtains when they are used as initial states is much slower. Hence, when starting with a squeezed state, there is a phase of rapid growth of the entropy which sends the system into a mixture of coherent states and which is followed by a period of slower increase. The entropy growth is in fact directly related to the decay of the squeezing. Since it is unlikely that the interactions in the early cosmology could be sufficiently weak so as to keep some squeezing, we can use the following mathematical result to infer that the actual entropy of the final distribution should be higher than (or equal to) a certain bound.

Coherent states indeed define a minimal decoherence scheme in the following sense. Consider the set of final distributions which result from the initial distribution of Eq. (33) through some decoherence process and which no longer possess any squeezed direction. The lowest value of the entropy in this set is given by the entropy of the incoherent superposition of coherent states, with statistical weights given by the probabilities to find the corresponding coherent states, as in Eq. (32). This distribution gives a lowest entropy simply because coherent states have minimal constant spreads (given quantum uncertainties and when considering free evolution, see Appendix A). We shall now explicitely write down this distribution and compute the entropy it carries.

C. Application to cosmology

When considering the initial distribution Eq. (33), one obtains the following density matrix:

$$\hat{\rho}_{min} = \int \frac{d^2v}{\pi}\frac{d^2w}{\pi}\, \mathcal{P}_{2,\,k}(v,w)\ |v,\, \mathbf{k}\rangle\langle v,\, \mathbf{k}| \otimes |w,\, -\mathbf{k}\rangle\langle w,\, -\mathbf{k}|\,, \tag{35}$$

where the probability distribution is given in Eq. (32). In Appendix B we show that Eq. (35) is indeed the resulting normalized distribution. The technical point which requires clarification is the extension of the above mentionned minimal scheme to two-mode squeezed states.

It should first be noted that when computing expectation values in leading order in n_k, the above distribution can be simplified and written as a single sum of two-mode coherent states, as in eq. (31):

$$\hat{\rho}_{min} \simeq \int \frac{d^2 v}{\pi} \left| \mathcal{A}_k(v) \right|^2 \, |v, \, \mathbf{k}\rangle\langle v, \, \mathbf{k}| \otimes |z v^*, \, -\mathbf{k}\rangle\langle z v^*, \, -\mathbf{k}| \, . \tag{36}$$

As we shall progressively see, this distribution should be conceived as the quantum counterpart of the effective distribution of sine functions discussed in section II.B.

Secondly by reducing the density matrix, some entropy has been introduced. One verifies indeed that $\mathrm{Tr}(\hat{\rho}_{min}^2) < 1$. The important point is that this decohered distribution has retained all the information about the temporal coherence of the modes. In fact one has

$$\begin{aligned} \mathrm{Tr}(\hat{\rho}_{min} \hat{a}_{\mathbf{k}} \hat{a}_{-\mathbf{k}}) &= \int \frac{d^2 v}{\pi} \frac{d^2 w}{\pi} \left[\mathcal{P}_{2,\,k}(v, w) \, v \, w \right] = \int \frac{d^2 v}{\pi} \left[\left| \mathcal{A}_k(v) \right|^2 v \, (z_k v^*) \right] = z_k |\alpha_k|^2 \, , \\ \mathrm{Tr}(\hat{\rho}_{min} \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}}) &= \mathrm{Tr}(\hat{\rho}_{min} \hat{a}^\dagger_{-\mathbf{k}} \hat{a}_{-\mathbf{k}}) = |\alpha_k|^2 = n_k + 1 \, . \end{aligned} \tag{37}$$

The first line shows that the cross term is equal to that of the original distribution, see Eqs. (15, 16). In the second line, one sees that the occupation numbers slightly differ, but to order $1/n_k$ only.

The effect of having increased by 1 the occupation number while having kept untouched the cross-correlation means two things. First, the (relative) degree of coherence has been reduced and therefore some entropy has been created. Secondly, in the high occupation limit, the two-point function of Eq. (18) is not affected by this loss of coherence: for relevant modes, the relative change being of the order of $1/n_k \sim 10^{-100}$.

D. Minimal entropy and the neglect of the decaying mode

The entropy of any Gaussian two-mode distribution can be exactly calculated [25, 26] by using the fact that the density matrix of a two-mode squeezed state is unitarily equivalent to the tensorial product of two thermal density matrices of oscillators, see Appendix C. We shall name a and b these two real oscillators. One has

$$\hat{\rho}_{2,\,k} = \mathcal{M}^\dagger \, \hat{\rho}_{th,\,a} \otimes \hat{\rho}_{th,\,b} \, \mathcal{M} \, , \tag{38}$$

where $\mathcal{M}$ is a unitary operator acting on the two-mode Hilbert space. The expression of the (von Neumann) entropy immediately follows:

$$S\left[\hat{\rho}_{2,\,k}\right] = S\left[\hat{\rho}_{th,\,a}\right] + S\left[\hat{\rho}_{th,\,b}\right] \, , \tag{39}$$

where the entropy of a thermal bath with mean occupation $\bar{n}$ is

$$S\left[\hat{\rho}_{th}\right] = (\bar{n} + 1) \ln(\bar{n} + 1) - \bar{n} \ln(\bar{n}) \, . \tag{40}$$

When considering only distributions preserving homogeneity and isotropy, the occupation numbers of the thermal matrices are equal and given by

$$\bar{n}_k + \frac{1}{2} = \left((n_k + \frac{1}{2})^2 - |c_k|^2 \right)^{1/2} , \tag{41}$$

where n_k and c_k are defined in Eq. (15).

Let us apply this result to several cases. First, for the two-mode in vacuum of Eq. (12), the occupation number and the coherence term are related by Eq. (16), one has $\bar{n}_k = 0$ as expected. Hence the entropy vanishes.

For the decohered matrix Eq. (35), using Eq. (37), the occupation number of the two thermal baths are

$$\bar{n}_k = \frac{1}{2} \left(-1 + \sqrt{8(n_k+1)+1} \right) \sim \sqrt{2 n_k} \, , \tag{42}$$

where the last term is the leading order when $n_k \gg 1$. The two-mode entropy of this mixture is then

$$\begin{aligned} S\left[\hat{\rho}_{min}\right] &= 2\, S\left[\hat{\rho}_{th}\right] = 2 \ln \bar{n}_k + O(1) \, , \\ &= \ln n_k + O(1) = 2\, r_k + O(1) \, , \\ &\simeq 100 \ln(10) \, . \end{aligned} \tag{43}$$

In the second line, we have expressed the occupation number in term of the squeezing parameter $r_k : \, n_k = \mathrm{sh}^2 r_k$. Hence, a two-mode squeezed vacuum state which decoheres in the two-mode coherent basis goes along with an entropy of $S_{\mathbf{k}, -\mathbf{k}} = 2\, r_k$ *per two-mode.* This value is large, but not maximal. Indeed, had the coherence term c_k vanished while keeping the occupation numbers fixed[7], one would have found the maximal value of the entropy which is given by twice this above value, i.e.

$$S^{max}_{\mathbf{k}, -\mathbf{k}} = S^{inc}_{\mathbf{k}, -\mathbf{k}} = 4\, r_k \, , \tag{44}$$

or $S^{inc}_{\mathbf{k}} = 2\, r_k$ per mode $\mathbf{k}$.

It is interesting to notice that the entropy associated with the Gaussian distribution Eq. (22) of sine functions equals that of Eq. (35), up to an arbitrary constant which arises from the usual ambiguity of attributing an entropy to a classical distribution. (This ambiguity can be lifted when introducing $\hbar$ to normalize the phase space integral.) Using this trick, the entropy associated with Eq. (22) is $S_{\mathit{eff}} = 2\, r_k$ for each *independent* mode, since this entropy is maximal. However, for each independent mode here means for each two-mode since the mode $-\mathbf{k}$ is no longer independent of the $\mathbf{k}$ mode once the cosines have been neglected, see Eq. (21). From this equality of entropies, we conclude that the quantum density matrix which corresponds to the Gaussian ensemble of sine functions is precisely given by Eq. (35).

A priory one might think that many quantum distributions can be associated with the classical distribution Eq. (22). This is not the case when imposing that Gaussianity is preserved and that the entropies coincide. Indeed, in the high squeezing limit, the entropy is an extremely sensitive function of the relative coherence. To see this dependence, let

us calculate the entropy of a generic distribution (15), and let us write the norm of the coherence term as

$$|c_k|^2 = n_k(n_k + 1 - \delta_k)\,, \tag{45}$$

where δ_k is a real number between 0 and $n_k + 1$. The equation (41) has the solution $\bar{n}(\bar{n}+1) = n_k\delta_k$. They are three characteristic values. $\delta_k = 0$ obviously corresponds to the pure squeezed state: the in vacuum. $\delta_k = 1$ corresponds to the minimal decoherence scheme with entropy given in Eq. (43). $\delta_k = n_k + 1$ corresponds to the thermal case with maximum entropy. From this analysis we see that the uncertainty in the definition of the quantum distributions which give rise to the entropy $S_{\mathbf{k},-\mathbf{k}} = 2r_k$ is very limited: δ_k must be of order 1 in the following sense. Consider that the loss of coherence scales as $\delta_k \propto n_k^\gamma$. Then the thermal occupation number and the entropy respectively scale as $\bar{n}_k \propto n_k^{(1+\gamma)/2}$ and $S_{\mathbf{k},-\mathbf{k}} = (1+\gamma)2r_k + Const$. This linear dependence in r_k implies that the distributions with entropy given by Eq. (43) all have $\gamma = 0$.

We re-emphasize that a value of δ_k smaller than 1 is unlikely in the context of primordial fluctuations since it would mean that the distribution has kept some of its quantum squeezeness. The remaining question thus concerns the computation of δ_k, noticing that it can receive contributions both from the inflationary period and from the adiabatic era[13]. The challenge is to determine which one is more important and what could be a realistic value of δ_k.

As a final comment, we provide an upper bound for the decoherence entropy which could have resulted from processes in the inflationary phase. Because increasing the decoherence implies increasing the power of the growing mode, one obtains a bound on the decoherence level when requiring that the power of the decaying mode be equal to that of the growing mode at the onset of the adiabatic era. (This requirement follows from the fact that the rms value of the primordial fluctuations (of the Bardeen potential) cannot be much higher than that obtained from in vacuum because otherwise this would invalidate the whole framework of linear metric perturbations.) Using Eq. (18) evaluated at η_r and the parameterization of Eq. (45), one obtains

$$\delta_k = n_k^{1/2}\,, \qquad S^{upper}_{\mathbf{k},-\mathbf{k}} = 3r_k + O(1)\;. \tag{46}$$

If no further decoherence is added in the adiabatic area, this should be the maximum amount of entropy stored in the primordial spectrum. Notice that when evaluated at recombination, the two-point function is still unaffected by this modification of the coherence because at that time the decaying mode has still further decreased. Indeed the residual modifications are then of the order of $n_k^{-1/2} \sim 10^{-50}$.

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APPENDIX A: COHERENT STATES

This appendix aims to present the properties which we shall use in the body of the manuscript. For more details, we refer to [28–30].

Coherent states (of a real oscillator) can be defined as eigenstates of the annihilation operator:

$$\hat{a}|v\rangle = v|v\rangle\,, \tag{A1}$$

where v is a complex number. In Fock basis it is written as

$$|v\rangle = e^{-\frac{|v|^2}{2}}\sum_{n=0}^{\infty}\frac{v^n}{\sqrt{n!}}|n\rangle\,, \tag{A2}$$

where the exponential prefactor guarantees that the state is normalized to unity $\langle v|v\rangle = 1$. They are also obtained by the action of a displacement operator on the vacuum :

$$|v\rangle = \hat{D}(v)|0\rangle = e^{v^*\hat{a}-v\hat{a}^\dagger}\,. \tag{A3}$$

The first interesting property of coherent states is that they correspond to states with a well defined complex amplitude v. Indeed, by definition (A1), the expectation values of the annihilation and creation operators are

$$\langle v|\hat{a}|v\rangle = v\,, \quad \langle v|\hat{a}^\dagger|v\rangle = v^*\,. \tag{A4}$$

Thus the mean occupation number is

$$\langle v|\hat{a}^\dagger\hat{a}|v\rangle = |v|^2\,. \tag{A5}$$

It is to be also stressed that the variances vanish:

$$\Delta\hat{a}^2 = \langle v|\hat{a}^2|v\rangle - \langle v|\hat{a}|v\rangle^2 = 0\,, \quad \Delta\hat{a}^{\dagger\,2} = \langle v|\hat{a}^{\dagger\,2}|v\rangle - \langle v|\hat{a}^\dagger|v\rangle^2 = 0\,. \tag{A6}$$

From these properties one sees that the expectation values of the position and momentum operators (in the Heisenberg picture)

$$\hat{q}(t) = \sqrt{\frac{\hbar}{2\omega}}\left(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right)\,, \qquad \hat{p}(t) = -i\sqrt{\frac{\hbar\omega}{2}}(\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t})\,,$$

are

$$\begin{aligned}\bar{q}(t) &= \langle v|\hat{q}(t)|v\rangle = \sqrt{\frac{\hbar}{2\omega}}(ve^{-i\omega t} + v^* e^{i\omega t}) = \sqrt{\frac{2\hbar}{\omega}}|v|\cos(\omega t - \phi_v)\,,\\ \bar{p}(t) &= \langle v|\hat{p}(t)|v\rangle = -i\sqrt{\frac{\hbar\omega}{2}}(ve^{-i\omega t} - v^* e^{i\omega t}) = -\sqrt{2\hbar\omega}|v|\sin(\omega t - \phi_v) = \partial_t\bar{q}(t)\,.\end{aligned} \tag{A7}$$

We have used the polar decomposition $v = |v|e^{i\phi_v}$. These expectation values have a well defined amplitude and phase and follow a classical trajectory of the oscillator. This is due to the "stability" of coherent states under the evolution of the free Hamiltonian $2H_0 = p^2 + \omega^2 q^2$: if the state is $|v\rangle$ at time t_0, one immediately gets from (A2) that at a later time t, the state is given by $|v(t)\rangle = |ve^{-i\omega(t-t_0)}\rangle$. Notice that the variances of the position and the momentum are

$$\Delta\hat{q}^2 = \frac{\hbar}{2\omega}\,, \qquad \Delta\hat{p}^2 = \frac{\hbar\omega}{2}\,. \tag{A8}$$

They minimize the Heisenberg uncertainty relations and are time-independent. Hence, in the phase space (q, p), a coherent state can be considered as a unit quantum cell $2\pi\hbar$ in physical units (see also (A14) for the measure of integration over phase space) centered on the classical position and momentum of the harmonic oscillator $(\bar{q}(t), \bar{p}(t))$. In the large occupation number limit $|v| \gg 1$, coherent states can therefore be interpreted as classical states since $\Delta\hat{q}/\sqrt{\bar{q}^2 + \bar{p}^2/\omega^2} = \Delta\hat{p}/\sqrt{\omega^2\bar{q}^2 + \bar{p}^2} = 1/2|v|$. This is a special application of the fact that coherent states can in general be used to define the classical limit of a quantum theory, see [30] and references therein.

One advantage of coherent states [28] is that the calculations of Green functions resembles closely to those of the corresponding classical theory (i.e. treating the fields not as operators but as c-numbers) provided either one uses normal ordering, or one considers only the dominant contribution when $|v| \gg 1$. We compute the Wightman function in the coherent state $|v\rangle$

$$\begin{aligned} \widetilde{G}_v(t,t') &= \langle v|\hat{q}(t)\hat{q}(t')|v\rangle \\ &= \langle : \hat{q}(t)\hat{q}(t') :\rangle_v + \frac{\hbar}{2\omega} e^{i\omega(t-t')}\,, \end{aligned} \tag{A9}$$

where we have isolated the contribution of the vacuum. The normal ordered correlator is order $|v|^2$:

$$\begin{aligned} \langle : \hat{q}(t)\hat{q}(t') :\rangle_v &= \frac{\hbar}{\omega} Re\left[\langle \hat{a}^2\rangle_v e^{-i\omega(t+t')} + \langle \hat{a}^\dagger\hat{a}\rangle_v e^{i\omega(t-t')}\right] \\ &= \frac{2\hbar}{\omega}|v|^2 \cos(\omega t - \phi_v)\cos(\omega t' - \phi_v) = \bar{q}(t)\,\bar{q}(t')\,. \end{aligned} \tag{A10}$$

We see that the perfect coherence of the state, namely $|\langle \hat{a}\hat{a}\rangle_v| = \langle \hat{a}^\dagger\hat{a}\rangle_v$ is necessary to combine the contributions of the diagonal and the interfering term so as to bring the time-dependent classical position $\bar{q}(t)$ in Eq. (A10).

The wave-function of a coherent state in the coordinate representation is given by

$$\psi_v(q) = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{\omega}{2\hbar}(q-\bar{q})^2 - i\frac{\bar{p}q}{\hbar}\right)\,, \tag{A11}$$

where $v = (\omega\bar{q} + i\bar{p})/\sqrt{2\omega\hbar}$. This follows from the definition $\langle q|\hat{a}|v\rangle = v\langle q|v\rangle$. From this equation one notes that two coherent states are not orthogonal. The overlap between two coherent states is

$$\langle v|w\rangle = \exp\left(v^* w - \frac{1}{2}|v|^2 - \frac{1}{2}|w|^2\right)\,. \tag{A12}$$

Nevertheless they form an (over)complete basis of the Hilbert space in that the identity operator in the coherent state representation $\{|v\rangle\}$ reads

$$\mathbf{1} = \int \frac{d^2v}{\pi}\, |v\rangle\langle v|\,. \tag{A13}$$

The measure is

$$\frac{d^2v}{\pi} = \frac{d(\mathrm{R}ev)d(\mathrm{I}mv)}{\pi} = \frac{d\bar{q}d\bar{p}}{2\pi\hbar}\,. \tag{A14}$$

The representation of identity can be established by calculating the matrix elements of both sides of the equality in the coordinate representation $\{|q\rangle\}$, with the help of (A11).

APPENDIX B: APPLICATION OF ZUREK & AL. RESULTS TO THE COSMOLOGICAL CASE

In this appendix we show that Eq. (33) is indeed the minimal decohered distribution by decomposing the complex mode $\hat{\phi}_{\mathbf{k}}$ into two real oscillators $\hat{\phi}_1$ and $\hat{\phi}_2$ given by its real and imaginary parts. Since the two-mode Hamiltonian is Hermitian, it splits into the sum of two identical one-mode oscillator Hamiltonians for $\hat{\phi}_1$ and $\hat{\phi}_2$ separately. Notice that the annihilation operators of these two real oscillators, $\hat{a}_1 = (\hat{a}_{\mathbf{k}} + \hat{a}_{-\mathbf{k}})/\sqrt{2}$, $\hat{a}_2 = -i\,(\hat{a}_{\mathbf{k}} - \hat{a}_{-\mathbf{k}})/\sqrt{2}$, mix $\mathbf{k}$ and $-\mathbf{k}$ annihilation operators. Hence they can easily take into account the entanglement between these two modes.

A two-mode squeezed state $|0in, \mathbf{k}\rangle_2$ can always be written as the tensorial product of the two one-mode squeezed states[31]. In our case, the one-mode squeezed states are those of the oscillators 1 and 2 because the Hamiltonian separates. Thus we have

$$|0in, \mathbf{k}\rangle_2 = |0in, 1\rangle \otimes |0in, 2\rangle \,. \tag{B1}$$

The one-mode squeezed states are governed by the same parameter $z/2$:

$$|0in, 1\rangle = \frac{1}{\sqrt{|\alpha|}} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n \frac{\sqrt{2n!}}{n!} |2n,\, 1\rangle \,. \tag{B2}$$

The same expression holds for the ket $|0in, 2\rangle$.

The overlap of this one-mode squeezed state with a one-mode coherent state is

$$\langle v,\, 1|0in, 1\rangle = \frac{1}{\sqrt{|\alpha|}} \exp\left(-|v_1|^2/2 + z v_1^{*\,2}/2\right) \,. \tag{B3}$$

According to [5], when taking small interactions into account, the density matrix of a one-mode squeezed state will preferable decohere into the mixture

$$\hat{\rho}_{red,\,1} = \int \frac{d^2 v_1}{\pi}\, \mathcal{P}_1(v_1)\, |v_1\rangle\langle v_1| \,. \tag{B4}$$

where the statistical weight is given by the probability to find a coherent state starting from the in vacuum :

$$\begin{aligned} \mathcal{P}_1(v_1) = |\langle v,\, 1|0in, 1\rangle|^2 \;&=\; \frac{1}{|\alpha|} \exp\left(-|v_1|^2 + Re(z v_1^{*\,2})\right) \,, \\ &=\; \frac{1}{|\alpha|} e^{-2R_1^2}\, e^{-I_1^2/2|\alpha|^2 + O(n^{-3/4} R_1 I_1)} \,. \end{aligned} \tag{B5}$$

In the second line we have introduced the real and imaginary parts of v_1 in order to show that one gets an ellipse of great axis equal to $|\alpha|^2$ which is oriented along the imaginary axis. The width of the small axis is $1/2$, as in vacuum.

For this decoherence procedure to be valid, as noticed in [12], it is important that the interactions do not break the coherence between $\mathbf{k}$ and $-\mathbf{k}$ modes, or equivalently do not mix ϕ_1 and ϕ_2.

The product of two one-mode coherent states 1 and 2 is also the product of a coherent state for the $\mathbf{k}$ and $-\mathbf{k}$ modes:

$$
\begin{aligned}
|v_1\rangle \otimes |v_2\rangle &= \hat{D}_1(v_1)\hat{D}_2(v_2)|0in,1\rangle \otimes |0in,2\rangle \,, \\
&= \hat{D}_{\mathbf{k}}(v)\hat{D}_{-\mathbf{k}}(w)|0in,\mathbf{k}\rangle \otimes |0in,-\mathbf{k}\rangle \,, \\
&= |v,\mathbf{k}\rangle \otimes |w,-\mathbf{k}\rangle \,,
\end{aligned} \tag{B6}
$$

where the amplitudes are related by

$$
v = \frac{v_1 + iv_2}{\sqrt{2}} \,, \qquad w = \frac{v_1 - iv_2}{\sqrt{2}} \,. \tag{B7}
$$

Finally, the product of the probabilities (B5) give the probability (32). Performing the change of variables from $(v_1,\, v_2)$ to $(v,\, w)$ completes the proof.

APPENDIX C: DIAGONALIZATION OF THE COVARIANCE MATRIX

Introducing the position and momentum variables for each mode, i.e. $\hat{a}_{\mathbf{k}} = (\hat{q}_{\mathbf{k}} + i\hat{p}_{\mathbf{k}})/\sqrt{2}$ and $\hat{a}_{-\mathbf{k}} = (\hat{q}_{-\mathbf{k}} + i\hat{p}_{-\mathbf{k}})/\sqrt{2}$, and defining the vector

$$
\zeta^\dagger = (\hat{q}_{\mathbf{k}}\ \hat{p}_{\mathbf{k}}\ \hat{q}_{-\mathbf{k}}\ \hat{p}_{-\mathbf{k}}) \,, \tag{C1}
$$

one has the covariance matrix

$$
C = \langle [\,\zeta_i,\, \zeta_j]_+ \rangle = \begin{pmatrix} n_k + \frac{1}{2} & 0 & c_r & c_i \\ 0 & n_k + \frac{1}{2} & c_i & -c_r \\ c_r & c_i & n_k + \frac{1}{2} & 0 \\ c_i & -c_r & 0 & n_k + \frac{1}{2} \end{pmatrix} , \tag{C2}
$$

where $[\,,\,]_+$ is the anticommutator. Notice that $\left(n_k + \frac{1}{2}\right)^2 - |c_k|^2 > 0$ is a necessary condition for the matrix to have positive eigenvalues.

The transformation Eq. (38) amounts to diagonalize this matrix:

$$
C = M^t T M \,, \tag{C3a}
$$

$$
T = \left(\bar{n}_k + \frac{1}{2}\right) \mathbf{1} \tag{C3b}
$$

The matrix T is the covariance matrix of the two thermal density matrices $\hat{\rho}_{th,\,a} \otimes \hat{\rho}_{th,\,b}$ in Eq. (38). The matrix M is the product of two local transformations and one global rotation R. The latter brings the covariance matrix C under a 2×2 bloc diagonal form. A product of local rotations $R_1(\theta_1) \oplus R_2(\theta_2)$ diagonalize each bloc, and the product of local squeezing $S_1(r_1) \oplus S_2(r_2)$ brings the resulting matrix under the form T. Explicitly one has

$$
\begin{aligned}
R(\phi) &= \begin{pmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & \cos\phi & 0 & -\sin\phi \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & \sin\phi & 0 & \cos\phi \end{pmatrix} , \\
R(\theta) &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} , \qquad S(r) = \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix} ,
\end{aligned} \tag{C4}
$$

with the rotation angles $\phi = \pi/4$, $\theta_1 = \theta_2$ given by $\tan 2\theta = -c_i/c_r$, and the squeezing parameter $r_1 = -r_2$ defined by $\mathrm{th} r = -|c_k|/(n_k + 1/2)$.

The eigenvalue $\bar{n}_k$ of the thermal matrices is easily obtained by conservation of the determinant:

$$\det C \;=\; \left((n_k + \frac{1}{2})^2 - |c_k|^2 \right)^2 = \left(\bar{n}_k + \frac{1}{2} \right)^4 .$$

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